FUNCTIONAL EQUATION FOR L-FUNCTIONS

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In this talk, we will discuss the functional equation for L-functions and its proof, which is Theorem 2.4.5 in *Automorphic Forms* by Anton Deitmar. But to understand L-functions, we first need to understand modular forms: what are they? For any $S \subset SL_2(\mathbb{Z})$, the *modular form (of level S) and weight* k is a function $g : \mathbb{H} \to \mathbb{C}$ which is analytic, automorphic, and "grows slow enough." The automorphy condition states that the modular form satisfies $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$. The growth condition states that $f(z)$ is bounded for any $\gamma \in SL_2(\mathbb{Z})$ as $\Im(z) \to \infty$. Now we can define the L-function attached to f.

Definition 0.1. Suppose the Fourier expansion of a *cusp form f of weight* k is given by $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ (a cusp form is a special modular form which has $a_0 = 0$.) Then the *L-function attached to* f is $L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ (for $s \in \mathbb{C}$), which converges locally uniformly in the region $\Re(s) > k/2 + 1$.

Why are L-functions important? We care about them because much interesting data is encoded in their metadata. For example, we can analyze their zeroes and poles to study arithmetic objects. Perhaps the most popular example is the fact that if $\zeta(1)$ (where ζ is an L-function!) is a pole, then there are an infinite number of primes.

Remark 0.2. For the Fourier coefficients a_n of a cusp form f of weight k (where k is an even natural at least 4), there exists $C > 0$ such that $|a_n| \leq C n^{k/2}$.

Definition 0.3. On the region $\Re(z) > 0$, the *Gamma function* is defined by the integral $\Gamma(z) = \int_0^{\infty} \frac{t^{z-1}}{e^t} dt$ and converges locally uniformly absolutely.

The following theorem illustrates a highly non-trivial fact.

Theorem 0.4. *Let f be a cusp form of weight k. Then,*

(1) *The L-function $L(f,s)$, which is initially holomorphic for $\Re(s) > k/2+1$, has an analytic continuation to an entire function. The extended function $\Lambda(f,s) := \frac{1}{(2\pi)^s}\Gamma(s)L(f,s)$ satisfies the functional equation*

$$\boxed{\Lambda(f,s) = (-1)^{k/2}\Lambda(f,k-s)}.$$

(2) *The function $\Lambda(f,s)$ is bounded on every vertical strip, viz. for every $T > 0$ there exists $C_T > 0$ such that $|\Lambda(f,s)| \leq C_T$ for every $s \in \mathbb{C}$ with $|\Re(s)| \leq T$.*

Proof. To be shown in-person (but here's an outline). □

The coefficients of the Fourier expansion of f by Remark 0.2 are bounded, so we can show f decreases rapidly as we move up the positive imaginary axis, viz. $|f(iy)| \to 0$ as $y \to \infty$. Likewise $y \mapsto \sum_{n=1}^{\infty} a_n e^{-2\pi y n}$ converges as $y \to \infty$. Thus $\int_{\varepsilon}^{\infty} f(iy)|y^{s-1}|dy$ converges absolutely so we can interchange sums and integrals, meaning the integral is equivalent to $\sum_{n=1}^{\infty} a_n \frac{1}{(2\pi n)^s} \int_{\varepsilon}^{\infty} \frac{y^{s-1}}{e^y} dy$. As $\varepsilon \to 0$, this expression converges to $\frac{1}{(2\pi)^s} L(f,s)\Gamma(s)$ for $\Re(s) > k/2+1$, which by definition is $\Lambda(f,s)$. *For boundedness*: Consider $\int_0^{\infty} f(iy)y^{s-1}dy = \int_0^1 + \int_1^{\infty}$. The left term Λ_2 trivially converges. The right term Λ_1 is entire because $f(iy) \to 0$ and $y \neq 0$, so we can show $\Lambda_1 \leq \int_1^{\infty} |f(iy)|y^{\Re(s)-1}dy$, which implies boundedness on vertical strips. *For functional equation:* Notice $f(i/y) = (yi)^k f(iy)$ where $(a,b;c,d) = (0,1;-1,0) \in SL_2(\mathbb{Z})$. Then this fact and the Mellin transform show the functional equation holds!

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References

[1] Anton Deitmar, Automorphic Forms, Springer, 2013.

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