Zeros of Period Polynomials of Bianchi Modular Forms

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Background: $SL_2(R)$ and the Upper Half Plane

$SL_2(R)$

$$SL_2(R) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathcal{R}, ad - bc = 1 \right\}$$

The Upper Half Plane

$\mathcal{H}_2 := \{z \in \mathbb{C} : Im(z) > 0\}$

The Action

let $g \in SL_2(\mathbb{R}), g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ We define the following action on the upper half plane: for $z \in \mathcal{H}_2$ we put

$$g(z) = \frac{az + b}{cz + d}$$

The Modular Group and the Upper Half Plane

The Modular Group

The group $\Gamma_1 \coloneqq \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{R})$ is called the *modular group*. Γ_1 is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, so the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}_2$ induces an action of Γ_1 on $\mathcal{H}_2$.

Generators of Γ

Let $S, T \in \Gamma$ defined respectively as $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. S and T generate Γ, i.e $\Gamma = \langle S, T : S^4 = I, (ST)^6 = I \rangle$.

The Action

The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}_2$ is generated by:
$S(z) = -\frac{1}{z}, T = z + 1$

The Upper Half Plane

$S(z) = -\frac{1}{z}$, $T = z + 1$

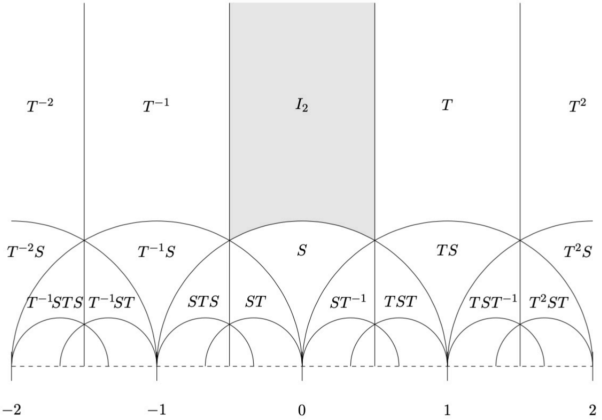


Figure: The upper half plane with the fundamental domain shaded gray.

Modular Forms

Definition: Classical Modular Forms

Given a finite index subgroup $\Gamma \subset \Gamma_1 = SL_2(\mathbb{Z})$, a *modular form* of level Γ and weight k is a holomorphic function

$$f : \mathcal{H}_2 \to \mathbb{C}$$

on the upper half plane $\mathcal{H}_2$ that satisfies the following property:

$$f(\gamma z) = (cz+d)^k f(z) \quad \forall z \in \mathcal{H}_2 \text{ and } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Generators

$f(Sz) = f(-\frac{1}{z}) = z^k f(z)$
$f(Tz) = f(z+1) = f(z)$

Bianchi

$\mathbb{Q}[i]$ and the Gaussian integers

$\mathbb{Q}[i] := \{a + bi : a, b \in \mathbb{Q}\}$.
The subring $\mathbb{Z}[i] \subset \mathbb{Q}[i]$ is called the Gaussian Integers. It is a natural extension of $\mathbb{Z} \subset \mathbb{Q}$.

K and $\mathcal{O}_K$

Let $K = \mathbb{Q}\left[\sqrt{-n}\right]$ for $n \in \mathbb{N}$ We call the discrete subset $\mathcal{O}_K \subset K$ the "ring of integers" of K.

Definition: Bianchi Modular Forms

Given a finite index subgroup $\Gamma \subset \Gamma_1 = \mathrm{SL}_2(\mathcal{O}_K)$, a *Bianchi modular form* of level Γ and weight k is a function

$$f : \mathcal{H}_3 \to \mathbb{C}^{k+1}$$

Period Polynomials

Modular Cusp Form

A *modular cusp form* is a modular form that vanishes at all its cusps.

Period Polynomial

A *period polynomial* is a polynomial containing values of an L-function associated with a modular cusp form.

Extending Pașol/Popa's results to the Bianchi case

Pașol/Popa [PP13] show that period polynomials of classical modular forms satisfy the period polynomial relations and they determine the space of classical period polynomials.

Using cohomology groups, Shapiro's Lemma, cocycles, and coboundries we are able to expand Pașol/Popa's results in the case of Bianchi modular forms and Bianchi period polynomials.

Extending Pașol/Popa's results to the Bianchi case

We found that a Bianchi period polynomial, ρ_F, satisfies the period polynomial relations:

$$\begin{aligned}
\rho_F|(1+S) &= \rho_F + \rho_F|_S &= 0\\
\rho_F|(1-L) &= \rho_F - \rho_F|_L &= 0\\
\rho_F|(1+U+U^2) &= \rho_F + \rho_F|_U + \rho_F|_{U^2} &= 0\\
\rho_F|(1+E+E^2) &= \rho_F + \rho_F|_E + \rho_F|_{E^2} &= 0
\end{aligned}$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $L = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $T_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$, $E = T_iSL$ and $U = TS$.

Since ρ_F satisfies the period polynomials relations, we have a description of the space of Bianchi period polynomials. These relations can be used to find trivial zeros of period polynomials.

Finding trivial zeros

Relations

Trivial zeros are zeros shared by every polynomial in the space.
We use the following relations, first given by [Kar21]

$$\left(P\Big|_{1+S}\right) : P(z_1, \overline{z_2}) + (z_1\overline{z_2})^k P\left(-\frac{1}{z_1}, -\frac{1}{\overline{z_2}}\right) = 0$$

$$\left(P\Big|_{1-L}\right) ::= P(z_1, \overline{z_2}) - P(-z_1, -\overline{z_2}) = 0$$

$$\left(P\Big|_{1+U+U^2}\right) : P(z_1, \overline{z_2}) + (z_1\overline{z_2})^k P\left(\frac{z_1 - 1}{z_1}, \frac{\overline{z_2} - 1}{\overline{z_2}}\right) + (z_1 - 1)^k(\overline{z_2} - 1)^k P\left(\frac{1}{1 - z_1}, \frac{1}{1 - \overline{z_2}}\right) = 0.$$

$$\left(P\Big|_{1+E_\omega+E_\omega^2}\right) : P(z_1, \overline{z_2}) + (z_1\overline{z_2})^k P\left(\frac{iz_1 + 1}{z_1}, \frac{-i\overline{z_2} + 1}{\overline{z_2}}\right) + (iz_1 + 1)^k(-i\overline{z_2} + 1)^k P\left(\frac{i}{iz_1 + 1}, \frac{i}{i\overline{z_2} - 1}\right) = 0.$$

(where we assume $\omega = i$ for the 4th relation) to find trivial zeros of the polynomial.

Finding trivial zeros

Using matrices to find more zeros

X	$\frac{-1}{X}$	$-X$	$\frac{X-1}{X}$	$\frac{1}{1-X}$
ϕ	$\overline{\phi}$	$-\phi$	$\overline{\phi}^2$	$-\phi$
$\overline{\phi}^2$	$-\phi^2$	$-\overline{\phi}^2$	$-\phi$	ϕ

$$\begin{matrix} \left(P\big|_{1+S}\right) \\ \left(P\big|_{1-L}\right) \\ \left(P\big|_{1+U+U^2}\right) \\ \left(P\big|_{1+S}\right) \\ \left(P\big|_{1-L}\right) \\ \left(P\big|_{1+U+U^2}\right) \end{matrix} \qquad \begin{bmatrix} 1 & \phi^{2k} & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & \phi^{2k} & \overline{\phi}^{2k} & 0 & 0 \\ 0 & 0 & 0 & 1 & \overline{\phi}^{2k} & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ \phi^{2k} & 0 & \overline{\phi}^{2k} & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} P(\phi,\phi) \\ P(\overline{\phi},\overline{\phi}) \\ P(-\phi,-\phi) \\ P(\overline{\phi}^2,\overline{\phi}^2) \\ P(-\phi^2,-\phi^2) \\ P(-\overline{\phi}^2,-\overline{\phi}^2) \end{bmatrix}$$

Finding trivial zeros

Trivial zeros found

- $(\pm 1, \pm 1)$
- $(\pm i, \pm i)$
- (ζ_6, ζ_6)
- (ζ_6^5, ζ_6^5)
- (ζ_6, ζ_6^5)*
- (ζ_6^5, ζ_6)*
- (ζ_3, ζ_3)
- (ζ_3^2, ζ_3^2)
- (ζ_3^2, ζ_3)*
- (ζ_3, ζ_3^2)*
- $(\zeta_{12}, \zeta_{12}^{11})$ **
- $(\zeta_{12}^{11}, \zeta_{12})$ **
- $(\zeta_{12}^7, \zeta_{12}^5)$ **
- $(\zeta_{12}^5, \zeta_{12}^7)$ **

- (ϕ, ϕ)
- $(-\phi, -\phi)$
- $(\overline{\phi}, \overline{\phi})$
- $(\overline{\phi}^2, \overline{\phi}^2)$
- $(-\phi^2, -\phi^2)$
- $(-\overline{\phi}^2, -\overline{\phi}^2)$
- $(i\phi, i\phi)$ **
- $(-i\phi, -i\phi)$ **
- $(i\overline{\phi}, i\overline{\phi})$ **
- $(i\overline{\phi}^2, i\overline{\phi}^2)$ **
- $(-i\overline{\phi}, -i\overline{\phi})$ **

*when $3 \nmid k$

**when $K = \mathbb{Q}[i]$

Computationally solving for zeros

- We solved for all the zeros of weight 2 examples in Mathematica (still working on general case).
- Input: Solve[P[z1, z2] == 0, {z1, z2}, Complexes]
- Output: $\{(z_1, z_2)\}$ which consists of
  1. $= (z_{0_1}, z_{0_2})$,
  2. $= (-\overline{z_{0_1}}, -\overline{z_{0_2}})$,
  3. $(z_1, g_1(z_1))$, or
  4. $(z_1, g_2(z_1))$.

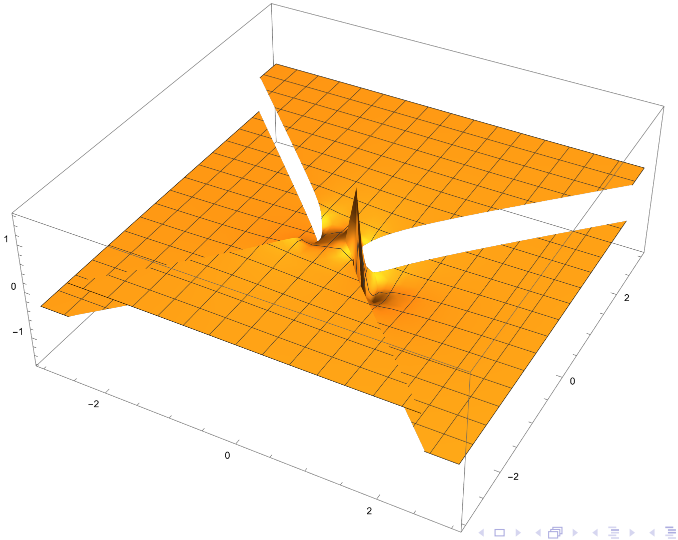
  where $g_1(z_1)$ and $g_2(z_1)$ are the positive and negative parts of

$$\frac{c_1 i + c_2 z_1 - c_3 i z_1^2 \pm c_4 \sqrt{(c_5 + c_6 i) - (c_7 + c_8 i) z_1 - (c_9 + c_{10} i) z_1^2 + (c_{11} + c_{12} i) z_1^3 + c_{13} z_1^4}}{c_{14} + c_{15} i z_1 + c_{16} z_1^2}$$

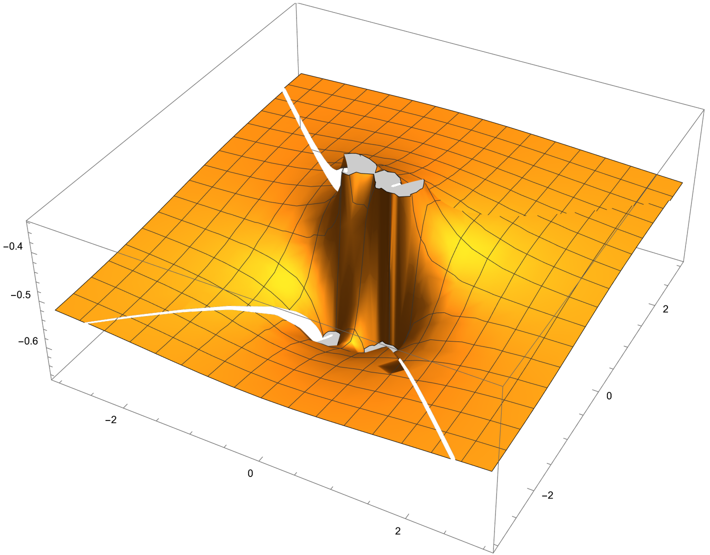
  for $c_j \in \mathbb{R}$ and $|z_{0_1}|, |z_{0_2}| < 1$.
- Following images correspond to Bianchi modular form 65.2-a over $\mathbb{Q}(\sqrt{-1})$ on LMFDB.

Real part of $g_1(z)$

$g_1(1+i) = 0.3 - 0.4i \implies (1+i, 0.3-0.4i)$ is a zero of P.

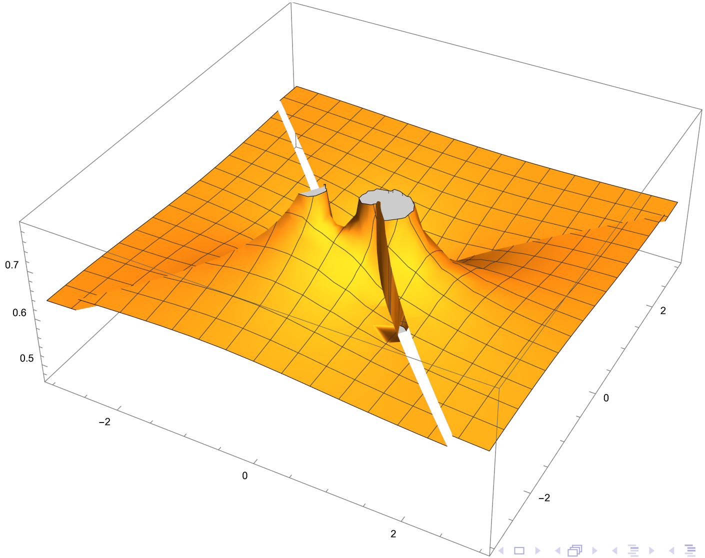


Imaginary part of $g_1(z)$



Norm of $g_1(z)$

$g_1(50i) = 0.354 - 0.393i$ and $g_1(1000i) = 0.354 - 0.3937i$.



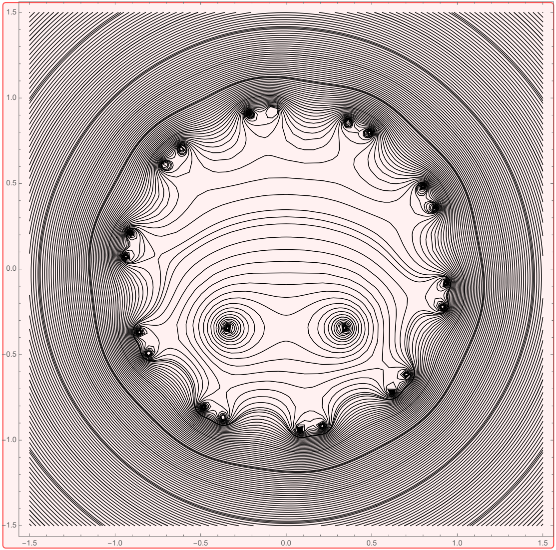
Zeros of the form (z, z^n) for $n \in \mathbb{Z}$

- The period polynomial P has $2 + 2|n|$ zeros of the form (z, z^n).
- All but 2 zeros lie on the unit circle.
- For $n \geq 3$, there are exactly nine terms in the polynomial $f(z, z^n)$. Explicitly P is of the form

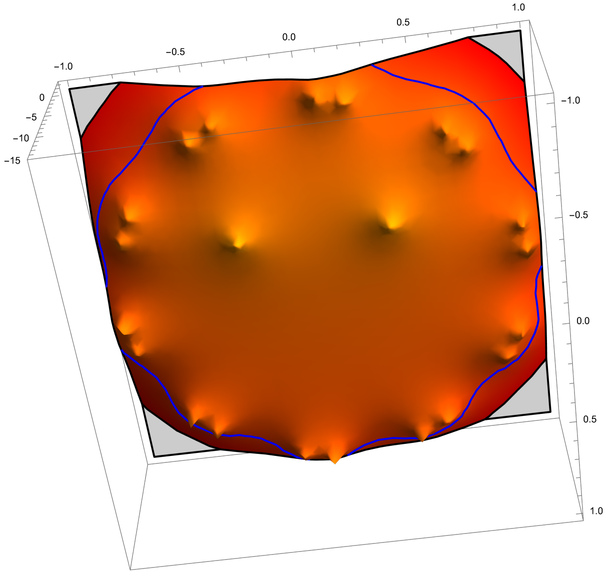
  $$c_0 + c_1 iz + c_2 z^2 + c_3 iz^n + c_4 z^{n+1} + c_5 iz^{n+2} + c_6 z^{2n} + c_7 iz^{2n+1} + c_8 z^{2n+2}$$

  where $c_0, \ldots, c_8 \in [0, 1]$.
- Examples: $n = 10, -5$.

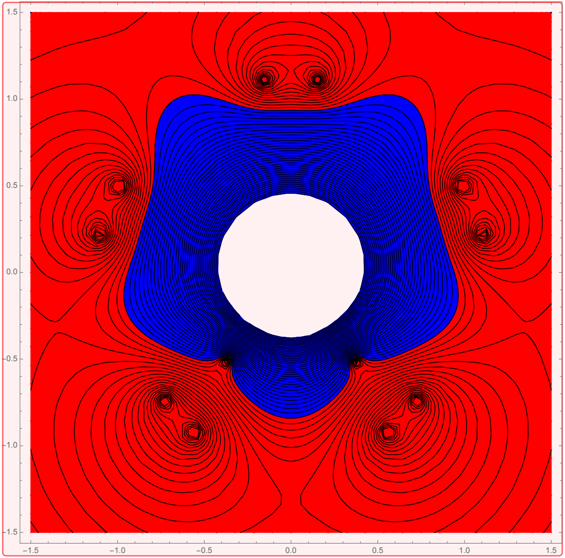
Contour plot of $\text{Log}|P(z, z^{10})|$



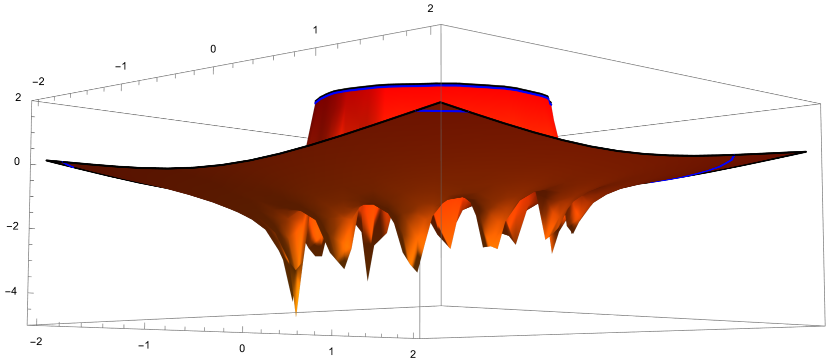
3D plot of $\mathrm{Log}|P(z, z^{10})|$



Contour plot of $\text{Log}|P(z, z^{-5})|$



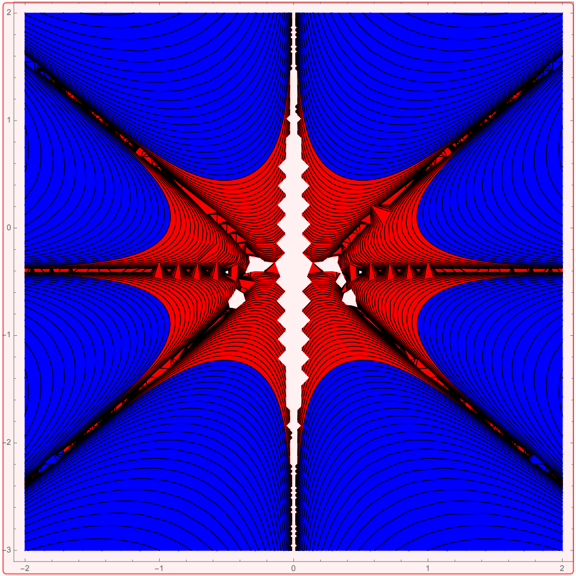
3D plot of $\text{Log}|P(z, z^{-5})|$



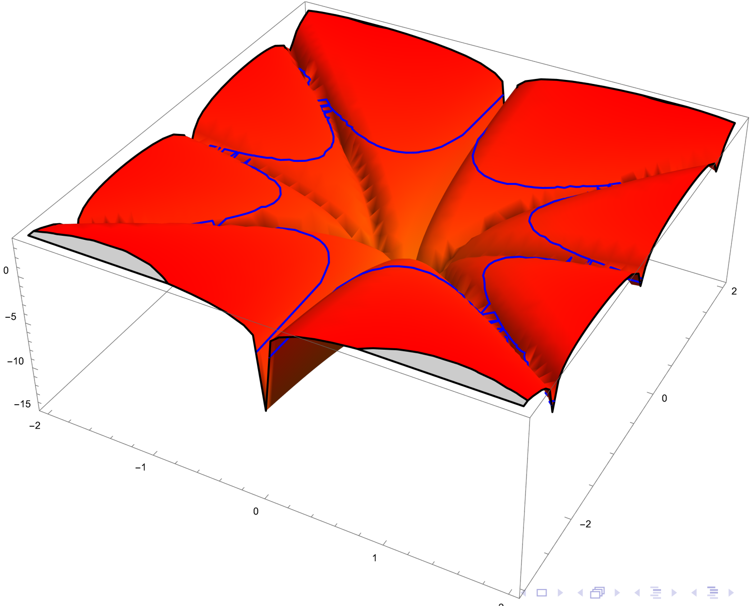
Additional Observations

- Number and location of zeros seem relatively consistent across different levels (although not necessarily across different underlying base fields).
- $|P(z,z)| - |P(-\overline{z}, -\overline{z})|$ vanishes everywhere (since $\geq, \leq 0$ are both true everywhere), so $|P(z,z)| = |P(-\overline{z}, -\overline{z})|$ (*norms even with twist*).
- So norm at a tuple $(x+iy, x+iy)$ is equal to the norm of its reflection over the imaginary axis $(-x+iy, -x+iy)$.
- Although $|P(z,z) - P(-\overline{z}, -\overline{z})|$ does not vanish everywhere, the graphs nevertheless have symmetries.

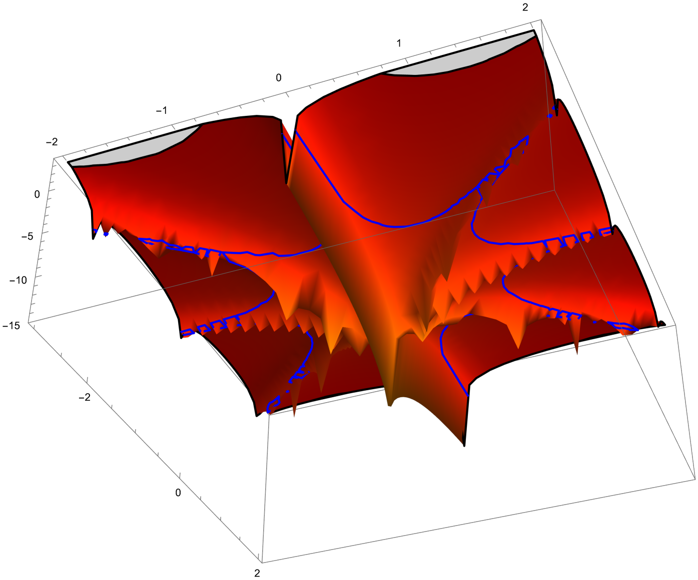
Contour plot of $\text{Log}\left|P(z,z) - P(-\bar{z},-\bar{z})\right|$



(Top view) 3D plot of Log$|P(z,z) - P(-\overline{z},-\overline{z})|$



(Bottom view) 3D plot of $\text{Log}|P(z,z) - P(-\overline{z}, -\overline{z})|$



Conclusion/Next Steps

- Solidify and prove claims which have strong computational support while acknowledging limitations in Mathematica.
- Generalizing Haberland's formula for the Bianchi case

Acknowledgments

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